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Motion on moduli spaces with potentials

Kasper Peeters

CERN, TH-division
1211 Geneva 23, Switzerland
E-mail: k.peeters@damtp.cam.ac.uk

Marija Zamaklar

DAMTP/CMS, Wilberforce road
Cambridge CB3 0WA, United Kingdom
E-mail: m.zamaklar@damtp.cam.ac.uk

Abstract: In the limit of small velocities, the dynamics of half-BPS Yang-Mills-Higgs solitons can be described by the geodesic approximation. Recently, it has been shown that quarter-BPS states require the addition of a potential term to this approximation. We explain the logic behind this modification for a larger class of models and then analyse in detail the dynamics of two five-dimensional dyonic instantons, using both analytical and numerical techniques. Non-zero modes are shown to play a crucial role in the analysis of these systems, and we explain how these modes lead to qualitatively new types of dynamics.

Keywords: Solitons Monopoles and Instantons, Brane Dynamics in Gauge Theories.

JHEP12(2001)032

Contents

1. Introduction

Much about the kinematics of supersymmetric solitons is known by virtue of the fact that their properties are related to the superalgebra of the underlying field theory. On the other hand the dynamical aspects of these objects are much harder to analyse. Extracting information about the dynamics of solitons by direct analysis of exact, time dependent solutions to the full equations of motion is generically too complicated. Instead, one uses the approximate but very powerful *moduli space technique* first introduced by Manton [1] for the study of the dynamics of four-dimensional Yang-Mills-Higgs magnetic monopoles.

Manton's method is based on the idea that the dynamics of solitons can be studied, to lowest order in the velocity, by a quasi-static approach. The set of static solutions, modulo small gauge transformations, is parametrised by a number of collective coordinates or moduli, and the space they span is called *moduli space*. At each moment of time the system is assumed to be infinitesimally close to a particular static configuration on this space. An effective action which tells us how the system evolves from one static configuration to another can be constructed as follows.

One first promotes the parameters in the static solution to be slowly varying functions of time. In order to deal only with the physical degrees of freedom, one has to ensure that this configuration, with time-dependent parameters, also satisfies the Gauss' constraint at the lowest order in the velocity and at each point in time. Very often this requires adding

JHEP12(2001)032

compensating gauge transformation terms to the static solution. This time-dependent configuration does by construction not change the potential energy, and the deviation from the original static solution is therefore called a *zero-mode*. If the initial (static) configuration was minimising the potential energy, then the zero-modes are deformations which will make the slowly moving system evolve through the set of states with the same, minimal value of the potential energy.

By inserting now this time-dependent solution into the action, expanding it to lowest order in the velocity and integrating over the space-like coordinates, one arrives at an effective action for the moduli of the static configuration. This action is just an ordinary non-linear world-line sigma model with the moduli space as the target space. The metric on it is naturally induced from the field theory action,[1] and the dynamical evolution of the system is described by geodesics on the moduli space. Although strictly speaking this approach is valid only in the limit of very small velocities, an analysis of the dynamics of two Yang-Mills-Higgs monopoles by Manton and Samols [3] shows that this method is surprisingly accurate. Radiation effects contribute only a tiny correction at speeds up to a considerable fraction of the speed of light.

An interesting feature that the moduli space method has brought to attention is the fact that four-dimensional 1/2-BPS *dyonic* solitons can be interpreted as zero-mode excitations of an underlying purely magnetically charged soliton. In particular the Julia-Zee dyon [4] arises naturally in this approach as an excited 't Hooft-Polyakov monopole. This can be seen by expanding the Julia-Zee field theory solution to lowest order in the electric charge. One is then left with an *unmodified* 't Hooft-Polyakov monopole and a non-vanishing A_0 component of the gauge field. Since the electric part of the energy of the dyon is a purely "kinetic" contribution to the total energy, the question arises as to what it is that is moving in these solutions. It turns out that the A_0 component of the solution is precisely equal to the global gauge rotation zero-mode of the starting 't Hooft-Polyakov monopole, with the velocity equal to the electric charge. The direction of this additional zero-mode is compact, which accounts for the quantisation of the electric charge. One concludes that the moduli space approximation for monopoles *automatically* incorporates dyons, and dynamical processes can lead to charge exchange.

Recent developments in the study of dualities in gauge theory and string theory have uncovered novel four and five-dimensional 1/4-BPS dyonic solutions, starting with the work of Bergman [5]. In contrast to the 1/2-BPS dyons discussed above, the dynamics of these solitons is naturally described on the moduli space of *approximately static configurations*. This is strongly tied to the way in which those solutions are constructed in the first place. In the cases known so far, the 1/4-BPS dyons can be seen as deformations of 1/2-BPS electrically uncharged solutions, obtained by turning on an additional Higgs field. The equations one has to solve are an *unmodified* BPS equation for the underlying 1/2-BPS

[1] Remarkably, the metric on the most studied example of two SU(2) monopoles was actually never obtained this way. Instead, Atiyah and Hitchin [2] were able to fix the metric completely using only symmetry arguments. This method has clear calculational advantages, as the zero-modes for this case are very complicated. In general situations, however, there are not enough symmetries and one has to resort to the zero-mode method described above.

JHEP12(2001)032

configuration, together with an equation for the additional scalar field in the background of the underlying 1/2-BPS soliton. The latter has a unique solution, and thus provides a one-to-one map between the moduli spaces of the two types of solitons. However, turning on the additional Higgs makes the system unstable, because the Higgs field introduces a new attractive force which is unbalanced. By further adding an additional electric charge, a compensating force is introduced which prevents the configuration from collapsing. The resulting solution is dyonic in the sense that it carries both the charge of the original 1/2-BPS soliton as well as the electric charge introduced to prevent collapse.

The instability of the intermediate step, where the Higgs is turned on but the electric charge is still zero, can alternatively be seen through the appearance of a *potential on moduli space*. This potential can in principle be derived just like the metric part of the action, but most of what is known is actually based on supersymmetry. Since rather general arguments show that the potential has to be written as the square of a Killing vector on moduli space, the analysis of Alvarez-Gaumé and Freedman [6] can be used to argue that the orbits of the Killing vector have to preserve the three complex structures in order for it to be compatible with the four supersymmetries. This observation, due to Tong [7] and elaborated on in many other papers, has led to a determination of explicit potentials for several dyon systems in SU(3) Yang-Mills-Higgs. The general structure of the effective action, and its supersymmetric extension, has been derived in a series of papers by Bak et al. [8, 9] and Gauntlett et al. [10, 11].

The electric charge again arises from a dynamical effect on the moduli space, just as in the case of the Julia-Zee dyon. However, the charge deformation is now a zero-mode of an *approximately static* configuration. Even though the starting configuration is unstable, the system becomes stable once the electric charge zero-mode is added. In fact, this excited configuration corresponds to the *exact* 1/4-BPS solution. Heuristically, the stabilisation arises because the centrifugal force due to the motion of the system in the compact direction is balanced against the force produced by the potential. In summary, these new static 1/4-BPS dyons are identified with closed geodesics of constant energy on the moduli space of 1/2-BPS solitons with a potential.[2] The conserved quantity specifying different geodesics is the angular momentum, and its value is identified with the electric charge carried by the dyon.

Although the origin and the form of the potential are thus rather well understood, it is important to realise that inclusion of a potential leads one *beyond* the strict moduli space approximation. The difference becomes most apparent when one explicitly tries to analyse the *dynamics* of such models, and the present paper is an attempt to do so. The addition of the potential clearly only makes sense when its magnitude and slope are small compared to other scales in the problem. In this regime, the unstable configurations solve the equations of motion to lowest order in the Higgs expectation value, and small deformations of this approximate solution can be studied. The picture one should have in mind is very similar to the picture of the 1/2-BPS case: the system is restricted to

[2] For lack of a better alternative we keep referring to the solutions to the equations of motion as "geodesics". Of course these solutions are not necessarily paths of extremal length when the effect of the potential is included.

JHEP12(2001)032

move at the bottom of a potential valley, which is surrounded by steep mountains. The *key difference* is that in the 1/4-BPS case the valley is slightly tilted. The smallness of the slope is required to prevent the system from gaining enough kinetic energy to start climbing the mountains.

With such a tilted valley, the deformations one has to consider are not just the zero-modes (i.e. the modes which keep the potential energy constant) but also the destabilising modes (i.e. the modes which change the potential by moving slowly down the tilted valley). In general it is very difficult to include non-zero modes in the dynamics, because there is no simple way to determine which of the non-zero modes are most relevant. While a systematic approach has been described by Manton [12], there are not very many models for which this has been worked out in detail (see Manton and Merabet [13] for one application and further references). In the present case, however, one can argue that an appropriate set of non-zero modes consists of the lifted zero-modes of the underlying 1/2-BPS soliton.

Given this insight, one can study the explicit solutions to the equations of motion of the effective action, and the symmetries that govern them. We will show how the inclusion of non-zero modes sometimes leads to the appearance of several separated scales in the low-energy dynamics, a new feature which does not occur for the 1/2-BPS models. Instead of ignoring the non-zero modes, one can actually integrate them out explicitly, and the resulting dynamics is qualitatively different from the dynamics predicted by just the zero-modes. Throughout the paper we will use the explicit example of two dyonic instantons, which are solitons in $4+1$-dimensional Yang-Mills-Higgs theory. They have the advantage over other 1/4-BPS solitons that the metric on moduli space is not too complicated, and can be constructed explicitly from a paper by Osborn [14].

The plan of the paper is then as follows. First, we recall the details of 1/4-BPS (multi) dyonic solitons in $4+1$-dimensional Yang-Mills-Higgs theory. In section 2.2 we then exhibit the appearance of a potential on moduli space, illustrated by this particular example. The dynamics of a single dyonic instanton can be studied analytically, which we do in section 2.3. The main part of our paper, contained in section 3, is concerned with the study of the two-soliton sector. We first derive the metric and potential, and then perform a detailed analysis of the dynamics. Qualitatively new features are found in section 3.3 and the final section is concerned with the separation of scales in the low-energy theory. We end with some comments on work in progress concerning 1/4-BPS dyons in $3+1$ dimensions.

Two appendices have been added. One discusses technical details of the two-instanton moduli space, while the other one summarises our conventions on quaternions and lists a number of useful expressions involving them.

2. Dyonic instantons and moduli spaces

2.1 The static dyonic instanton solution

After the general introduction we will, in the rest of the paper, specialise to a concrete example, namely the study of dyonic instantons. Before we discuss their dynamics, let us first introduce the static solution. The model under consideration is the five-dimensional

Yang-Mills-Higgs action, given by

$$S = -\int \mathrm{d}^5x \left(\frac{1}{4}\,\mathrm{tr}\,(F_{\mu\nu}F^{\mu\nu}) + \frac{1}{2}\,\mathrm{tr}\,(D_\mu\Phi D^\mu\Phi)\right), \tag{2.1}$$

with $\mu = (0, (i = 1, \ldots, 4))$. This model can be extended to a supersymmetric one, but we will not need the details of this extension in the present paper.

For static solutions the equations of motion reduce to

$$D_j F^{ji} + [A_0, D^i A_0] - [\Phi, D^i\Phi] = 0\,, \tag{2.2}$$

$$D_j F^{j0} - [\Phi, D^0\Phi] = 0\,, \tag{2.3}$$

$$D_i D^i \Phi = 0\,. \tag{2.4}$$

The first line clearly shows that by choosing $A_0 = \pm\Phi$ the last two terms cancel and the remainder of the equation is solved by an (anti) self-dual field-strength.[3] One can alternatively see this by rewriting the hamiltonian in Bogomol'nyi form, which leads to the following BPS equations,

$$F_{ij} = s\frac{1}{2}\epsilon_{ijkl}F^{kl}\,, \tag{2.5}$$

$$F_{0i} = s' D_i\Phi\,, \tag{2.6}$$

with s and s' two arbitrary signs (corresponding to the sign of the instanton and electric charge, respectively). Combining the second BPS equation with the Gauss law and going to the $A_0 = \pm\Phi$ gauge one arrives again at the equation of motion (2.4) for the Higgs field, which is just a Laplace equation in the background of the instanton. Hence as advertised, the equations for the 1/4-BPS dyon reduce to two equations: a BPS equation for the instanton and a Laplace equation for the scalar. An important observation is that for each instanton solution to (2.5) there is a unique solution to (2.4). Hence there is *a one to one map between the moduli space of instantons and the moduli space of dyonic instantons.*

For instantons with identical group embedding angles, the solution to these equations is given by Lambert and Tong [15], extended by Eyras et al. [16] and given a brane interpretation by one of the authors in [17]. For $s = -1$ one obtains,

$$A_0 = s'\frac{v}{H}T^3\,, \qquad A_i = \eta^a_{ij}\,\partial_j(\ln H)\,T^a\,, \qquad \Phi = -\frac{v}{H}T^3\,, \tag{2.7}$$

where the function H is given by

$$H = 1 + \sum_i \frac{\rho_i^2}{|x^i - y^i|^2}\,. \tag{2.8}$$

The symbols η^a_{ij} are the 't Hooft symbols and the matrices T^a are Pauli matrices. This solution is characterised by two conserved charges: the instanton number

$$I = \frac{1}{2}\int \mathrm{d}^4x\,\epsilon^{ijkl}\,\mathrm{tr}(F_{ij}F_{kl}) = N\,, \tag{2.9}$$

[3] For Julia-Zee dyons all three terms mix and the relation between A_0 and Φ is less rigid.

and the electric charge

$$q = s' \frac{1}{v} \int \mathrm{d}^4 x \ \mathrm{tr}(D_i \Phi)^2 = 4\pi^2 v s' \sum_i \rho_i^2 \ . \tag{2.10}$$

This solution can be extended to an arbitrary ADHM configuration for the four-dimensional subspace, as the solution to the Higgs field in the instanton background has been derived in terms of ADHM data by Dorey et al. [18]. We will make use of this later when we study the two-dyon dynamics.

2.2 Modes, zero-modes and the moduli space approximation

We now want to analyse the dynamics of the dyonic instanton solutions. The solution which was described in the previous section is an *exact, static* solution of the equations of motion. In principle one could try to construct the zero-modes of this configuration and the corresponding effective action for them. This method was followed by Bak and Lee [19] for 1/4-BPS monopoles. Our approach is more in spirit of Manton's original treatment [20] of Julia-Zee dyons.[4] Namely, we will try to make a link between dyonic instantons and instantonic solitons (i.e. instanton solutions trivially embedded in the $4+1$-dimensional Yang-Mills theory) and profit from the vast knowledge about zero-modes of the latter.[5] As we have already mentioned, there is a one-to-one map between the moduli space of instantons and the moduli space of dyonic instantons. However, since the non-zero Higgs expectation value of a dyonic instanton can not be obtained as a (zero) mode deformation of the instanton, one cannot start from the instanton moduli space. Instead one takes the "instanton solution" in the Yang-Mills-Higgsed theory as a starting point.

It is well known that due to Derrick's theorem [21], turning on the scalar field in the four-dimensional *euclidean* Yang-Mills-Higgs theory destabilises the instanton solution against collapse: this theory does not admit finite action solutions with non-trivial scalar. The absence of *exact* solutions does not, however, imply that one cannot make use of *approximate* solutions of the field equations. These solutions are obtained by expanding in small expectation values of the Higgs field at infinity. The equations of motion for the four-dimensional euclidean theory are

$$D_i F^{ij} = [\Phi, D^j \Phi] \,, \tag{2.11}$$

$$D^2 \Phi = 0 \,. \tag{2.12}$$

To lowest order in the Higgs expectation value the right-hand side of eq. (2.11) can be set

[4] It is an interesting open problem to construct a moduli space effective action for two Julia-Zee dyons *starting* from the dyonic solution, and understand to what extent such an action agrees with the one constructed with monopoles as starting point. This would be the analogue of starting from two dyonic instantons in the model under consideration here.

[5] It easy to check that zero-modes of the instanton are zero-modes of the corresponding instantonic soliton.

JHEP12(2001)032

to zero and the solution is then given by (2.7) with $A_0 = 0$. This approximate solution trivially lifts to a solution of $4+1$-dimensional Yang-Mills-Higgs. We will refer to this lifted solution, perhaps slightly inadequate, as the "constrained instantonic soliton".[6]

Our starting point is thus the moduli space of constrained instantonic solitons. Different points in this set are field configurations of *different* potential energy which solve equation of motion only up to *quadratic* order in the Higgs expectation value. Hence, unlike the situation for 1/2-BPS dyons, an effective action that describes the time evolution from one point in moduli space to the other necessarily involves deformations which lead to a change in the potential energy i.e. deformations which are *not* zero-modes. We will refer to generic deformations that satisfy Gauss' law as *modes*, and to the subset of these which keep the potential fixed as *zero modes*.

Time evolution in the *full* field theory can be approximated by geodesic motion on the set of constrained instantons, provided the time evolution in the full field theory is such that a system which is initially tangent to this set stays near the set. In order for this to be true, three requirements have to be met: the valley should be completely known, the bottom of the valley has to be almost flat and the edges surrounding the valley should be steep. The first condition is necessary because there is in general no mechanism that restricts the motion to a subset of the valley (unless, of course, there are conservation laws at work). The second condition, together with the third, ensure that the static force is small, so that the system never develops enough kinetic energy to start climbing the mountains that surround the valley. Note that the whole construction is completely analogous to the one for 1/2-BPS dyons, with the exception that the valley is in that case precisely flat. In the general case, a slightly tilted valley is allowed. The fact that the set of constrained instantons is a complete set follows from the fact the scalar equation (2.12) admits only a single solution given a self-dual (i.e. potential minimising) gauge-field background.

So let us now analyse the modes of the constrained instanton. A subset of the zero-modes of the instantonic soliton will also be zero-modes of the constrained instantonic soliton, while other zero-modes will be lifted by the presence of the non-zero Higgs. However, all of these modes have to be taken into account since the deformation by both types leaves us on the set of constrained instantons. Instanton zero-modes are very well studied in the literature (for a review see for instance Belitsky et al. [23]). For the charge one SU(2) instanton, there are eight of them: four corresponding to a change in position, one associated to the change of the instanton size and three more having to do with the global orientation of the instanton in the gauge group. The associated moduli are y^i, ρ and the three angles θ^a ($a = 1, 2, 3$) which we have not considered so far. Our time-dependent ansatz is therefore taken to be of the form

$$A_0 = \left(\frac{\dot{\theta}^a}{H}\right) T^a + \dot{y}^i \eta^a_{ij}\, \partial_j (\ln H)\, T^a \,, \qquad A_i = \eta^a_{ij}\, \partial_j (\ln H)\, T^a \,, \qquad \Phi = -\frac{v}{H} T^3 \,, \quad (2.13)$$

[6]In the four-dimensional theory, these approximate solutions were used by Affleck [22] to systematically compute instanton corrections in the Higgs phase, despite the fact that an exact solution is not available. His method of fixing the instanton to a given size by hand is known as the "constrained instanton method", and the lowest order term is obtained in the way described above.

where the parameters ρ, y^i and θ now depend on time. The second term in A_0 can be deduced by using a Lorentz boost of the static solution, or alternatively one can see that it corresponds to the instanton translation zero-mode. The θ-dependent terms are the global gauge embedding zero-modes, gauge transformed so that they sit in A_0. When ρv and $\dot\rho$ are both small such that their square can be neglected, one now indeed verifies that (2.13) satisfies Gauss' law: $D_i\partial_0 A_i = 0$ by itself and $D_i D^i A_0 = 0$ by virtue of the static equation of motion. In general, we need $(\dot\rho)^2 \ll 1$, $(\rho\dot\theta)^2 \ll 1$ and $(\rho v)^2 \ll 1$ for the approximation to be valid (more about this issue in section 3.2).

As the translational zero-modes are relatively uninteresting for a single dyonic instanton, let us set $\dot y^i = 0$ and study the dynamics of the size and group embedding parameters. We insert the ansatz (2.13) into the action (2.1) and integrate over the space-like slices using the basic integral

$$I_0 = \int \mathrm{d}^4x \, \frac{\rho^2 r^2}{(r^2+\rho^2)^4} = \frac{1}{3}\pi^2 \,, \tag{2.14}$$

in terms of which we have

$$I_1 \equiv \int \mathrm{d}^4x \left(D_i\left(\frac{1}{H}\right)\right)^2 = 12\rho^2\, I_0 \qquad \text{and} \qquad I_2 \equiv \int \mathrm{d}^4x \, (\partial_\rho\partial_i \ln H)^2 = 16\, I_0 \,. \tag{2.15}$$

With $(F_{0i})^2 = 3\, I_2\, \dot\rho^2 + I_1\, \dot\theta^2$ and $(D_i\Phi)^2 = v^2\, I_1$, the resulting moduli space action is

$$S_{\text{moduli}} = 2\pi^2 \int \mathrm{d}t \left(4\,\dot\rho^2 + \rho^2(\dot\theta^a)^2 - v^2\rho^2\right), \tag{2.16}$$

so that $\theta^a \in [0, 4\pi)$. The A_0 component always leads to a kinetic energy contribution, and together with the time variation of A_i they build up the first two terms in the action. These coincide with those of the pure instanton. The third term, which is the most interesting one, arises because we are in the Higgs phase, and originates from the potential energy of the scalar.[7] Note that, in contrast to earlier work where potentials arose, we have derived it here simply by inserting the explicit Higgs field from (2.13) in the field theory action.

As our θ zero modes are only valid to lowest order in θ^a, the action above is only correct when the angles are small. For arbitrary angles, the factor $(\dot\theta^a)^2$ has to be generalised to the SU(2) group volume. From now on, we will for simplicity restrict our attention to a single angle θ corresponding to the U(1) subgroup selected by the Higgs value at infinity. This is a consistent truncation because one can check that it restricts to a geodesic submanifold. For this particular angle the kinetic term equals $\rho^2\dot\theta^2$ even for finite θ.

[7] The kinetic energy $\int (D_0\Phi)^2$ of the scalar field deserves special attention, as it diverges logarithmically. This is very similar to the situation for sigma model lumps (see Ward [24], Ruback [25] and Leese [26]). Given that it comes with a coefficient $v^2\dot\rho^2$, one might be tempted to interpret the divergence as a constraint on the dynamics of ρ. However, this would be incorrect for several reasons. First of all, the term comes in at higher order in the approximation, and could receive additional contributions. More importantly, we know that for vanishing electric charge, $\dot\rho = 0$ *cannot* be a solution to the equations of motion of the full theory, as it is forbidden by Derrick's theorem. This seems to suggest that in the approximation we make, this term can be ignored. Although a full resolution of this puzzle is definitely still lacking, even for sigma model lumps, we have verified that the cut-off method of Piette and Zakrzewski [27] indeed confirms the correctness of our procedure.

JHEP12(2001)032

The equations of motion of the model (2.16) then consist of a conservation law for *angular momentum*,

$$\dot{\theta}\,\rho^2 = L\,, \tag{2.17}$$

where L is an arbitrary constant, together with the evolution equation for the radius ρ,

$$-\,8\,\ddot{\rho} + 2\,\rho(\dot{\theta}^2 - v^2) = 0\,. \tag{2.18}$$

We will solve these in the general case in the next section, but there are two special solutions which deserve separate attention. In the first case we look at geodesics with non-zero angular momentum. By choosing the value of θ appropriately one can cancel the potential term completely. In this case, (2.17) fixes the radius to be constant under time evolution,

$$\dot{\rho} = 0\,, \qquad \dot{\theta} = \pm v\,. \tag{2.19}$$

This is the force balance discussed in the introduction. The geodesic rotation in the direction chosen by the Higgs corresponds to an exact dyonic instanton solution (with the signs corresponding to the freedom in choosing s' in (2.7)). The electric charge of dyonic instanton arises just as with the Julia-Zee dyon: it is associated to kinetic energy in a compact direction (and corresponds to the conserved angular momentum).

On the other hand, when the angular momentum is zero, the radius of the instanton is driven to zero by the presence of the potential, and to the extent to which this approximation is valid, one regains the shrinking behaviour of the instantonic soliton in the Higgs phase.[8]

2.3 Dynamics of a single dyonic instanton

The equations of motion (2.18) and (2.17) can be solved exactly, and as we will see, the single dyonic instanton already shows interesting dynamical behaviour, much richer than that encountered for 1/2-BPS solitons. Upon inserting the conservation law $L = \dot{\theta}\rho^2$ into the equation of motion for ρ we get

$$4\,\ddot{\rho} - \frac{L^2}{\rho^3} + v^2\rho = 0\,. \tag{2.20}$$

Multiplying by $\dot{\rho}$, this can be integrated once to

$$4\,(\dot{\rho})^2 + \frac{L^2}{\rho^2} + v^2\rho^2 = c^2\,. \tag{2.21}$$

Upon further substitution of $\rho^2 = x$ this becomes a standard integral,

$$\int \frac{\mathrm{d}x}{\sqrt{-v^2x^2 + c^2x - L^2}} = -\frac{1}{\sqrt{4L^2v^2 - c^4}}\arcsin\left(\frac{-2v^2x + c^2}{\sqrt{4L^2v^2 - c^4}}\right). \tag{2.22}$$

[8] Even in the pure Yang-Mills theory the instanton will start to shrink if it is given an initial $\dot{\rho} \neq 0$, since the equation of motion (2.18) reduces to $\ddot{\rho} = 0$. A numerical analysis of the shrinking behaviour in pure Yang-Mills was given by Linhart [28, 29], who showed that the moduli space approximation result $\rho = \rho_0 + \dot{\rho}_0\, t$ gets modifications in the full theory (the parameter ρ becomes a function of r and the time evolution is modified as well). A similar analysis could presumably be done in the Higgs phase, where corrections to the $4\,\ddot{\rho} = -v^2\rho$ evolution of the moduli space approximation are expected to be found.

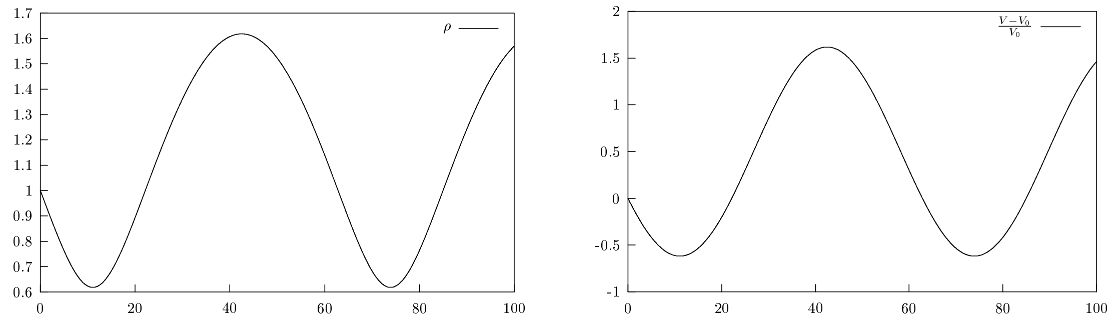

Figure 1: Typical oscillating orbit of a single dyonic instanton. The radius is shown in the first plot, while the second one exhibits the deviation of the potential from its initial value. Observe that the potential is not constant even though the solution remains completely within the regime of validity of the moduli space approximation. The initial data are $\rho(t=0)=1$ and $\dot{\rho}(t=0)=-0.05$, while the configuration is initially rotating in the T^3 direction with critical velocity, $\dot{\theta}=v$. Note that it is impossible to have a critically rotating system when ρ^2 goes through its equilibrium value; this follows directly by inserting $L=v\rho^2$ and $t=-t_0$ in (2.23).

The solution is thus a harmonic oscillator for ρ^2. Eliminating the constant c in favour of the amplitude by using (2.21), the expression for ρ is

$$\rho=\sqrt{A\sin\left(v(t+t_0)\right)+\sqrt{\frac{L^2}{v^2}+A^2}}=\sqrt{2A\sin^2\left(\frac{1}{2}v(t+\tilde{t}_0)\right)+\sqrt{\frac{L^2}{v^2}+A^2}-A}\,. \quad (2.23)$$

In the limit $L^2=0$ we thus recover the harmonic oscillator with frequency $\frac{1}{2}v$, as expected from (2.20). However, we should stress that as long as $\dot{\theta}\neq 0$ (or in other words, as long as $L^2\neq 0$), the oscillating solution described by (2.23) does not go through zero. It is thus fundamentally different from the pure constrained instantonic soliton: whereas the solution with $\dot{\theta}=0$ goes through the singular point $\rho=0$ in moduli space, and is therefore not necessarily a valid orbit for arbitrary times, the oscillations found above are genuine orbits that remain within the validity of the moduli space approximation. A typical orbit is depicted in figure 1 and compared with the 1/4-BPS dyonic instanton in figure 2.

It is also worth stressing that the generic solution does *not* stay at a fixed value of the potential. This is again different from the pure dyonic instanton with $\dot{\theta}=v$, which does conserve the potential and, because the potential gradient is exactly balanced by the centrifugal force, has all other parameters independent of time. The only quantity which is conserved in the general case is the angular momentum L, related to the electric charge of the system.

3. Dynamics of two dyonic instantons

3.1 Metric and potential on moduli space

With the understanding of the previous section at hand, we can now start the main part of our program and construct the moduli space approximation for the double dyonic instanton

JHEP12(2001)032

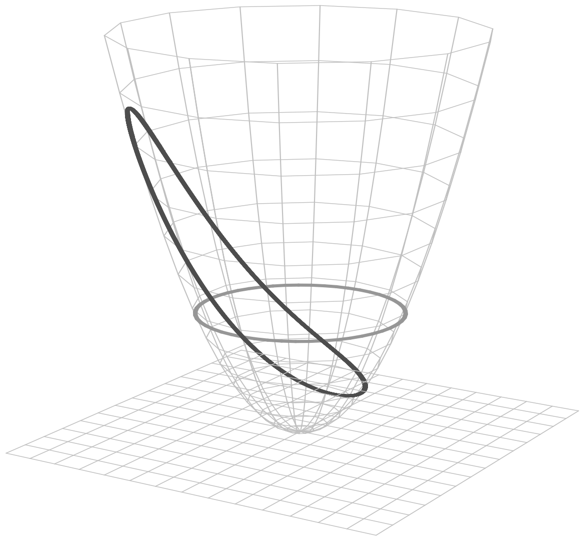

Figure 2: Moduli space representation of the intrinsic dynamics of a dyonic instanton. The x-y plane is the moduli space of the underlying instantonic soliton. The dyon is given by the circular orbit at fixed height of the potential, with the angular velocity being related to the electric charge (light curve). More kinetic energy can be added to make the system oscillate around the 1/4-BPS configuration (dark curve, corresponding to the data in figure 1).

system. The modes relevant for the computation of the metric are again the zero-modes of the two-instanton solution. The resulting metric can be obtained from the work of Osborn [14] (see also the work of Bruzzo et al. [30] and Bellisai et al. [31] who used the hyperKähler quotient construction). In addition, there will be a potential term coming from the scalar sector. We will treat these two parts of the calculation in turn.

Osborn's work [14] is focused on the calculation of the two-instanton metric determinant, but his paper contains enough intermediate results to extract the metric itself too. Details can be found in appendix A.1. Let us first consider the case when the two solitons are far apart. In this case the interpretation of the quaternions used in the ADHM construction (see appendix A.1) is straightforward: the magnitudes $|v_1|$ and $|v_2|$ are the sizes of the instantons, the unit normalised quaternions $v_1/|v_1|$ and $v_2/|v_2|$ correspond to the SU(2) angles, the centre of mass is ρ and the positions are given by $\rho+\tau$ and $\rho-\tau$, respectively. First, consider the limit $|\tau|\to\infty$. The metric then only receives contributions from the first four terms of (A.12). In vector notation, this reads

$$\mathrm{d}s^2 = 4\left(\mathrm{d}|v_1|\,\mathrm{d}|v_1| + |v_1|^2(\Omega_1)^2 + \mathrm{d}|v_2|\,\mathrm{d}|v_2| + |v_2|^2(\Omega_2)^2 + (\mathrm{d}\rho)^2 + (\mathrm{d}\tau)^2\right). \qquad (3.1)$$

This correctly reduces to two copies of the flat metric of the single dyonic instanton displayed in the first two terms of (2.16).

In case the group embedding angles are aligned the potential can be obtained from the 't Hooft ansatz for Φ,

$$\Phi = -\frac{v}{H}\,, \qquad H = 1 + \frac{\rho_1^2}{(r-r_1)^2} + \frac{\rho_2^2}{(r-r_2)^2}\,. \qquad (3.2)$$

The resulting potential has been computed by Eyras et al. [16],

$$V = -(\rho_1^2 + \rho_2^2) v^2 \,. \tag{3.3}$$

This is, as we will see, also the correct potential in the large separation limit, and the equations decouple completely. However, the full potential for the two-instanton case is more complicated and depends on the relative gauge orientation.

When taking into account the $1/(\text{distance})^2$ corrections to the metric, there are contributions from two terms in appendix A. Interestingly, there are no corrections that depend on the impact parameter (i.e. that contain the angles of the quaternion τ) at this order. Adding the second term in (A.14) and the first two terms (squared) in (A.20), and also setting the centre of mass coordinate to a constant, we get

$$\begin{aligned} \mathrm{d}s^2 =& \, 4\left((\mathrm{d}v_1)^2 + (\mathrm{d}v_2)^2 + (\mathrm{d}|\tau|)^2\right) + \\ &+ \frac{1}{|\tau|^2}\Big((\mathrm{d}v_2\mathrm{d}v_2)|v_1|^2 + (\mathrm{d}v_1\mathrm{d}v_1)|v_2|^2 + 2\,(v_1\mathrm{d}v_1)(v_2\mathrm{d}v_2) - (v_1\mathrm{d}v_2)^2 - \\ &\qquad - (v_2\mathrm{d}v_1)^2 - 2(v_1v_2)(\mathrm{d}v_1\mathrm{d}v_2) - \left((v_1\mathrm{d}v_2) - (v_2\mathrm{d}v_1)\right)^2\Big) \,. \end{aligned} \tag{3.4}$$

The term in (A.16) which contains an epsilon term will only contribute when the vectors v_1 and v_2 and their variations span the four-dimensional space, i.e. they contribute only to terms in the metric which involve all three angles. A consistent truncation is thus to set two of these angles identically zero, just as we did in the single instanton case in section 2.3. In this case, it is convenient to use the parameterisation

$$v_1 = \rho_1\left(\cos\theta_1 e_x + \sin\theta_1 e_y\right) \Longrightarrow \mathrm{d}v_1 = v_1(\theta_1)\,\frac{\mathrm{d}\rho_1}{\rho_1} + v_1\left(\theta_1 + \frac{\pi}{2}\right)\mathrm{d}\theta_1 \,, \tag{3.5}$$

and similarly for v_2. The metric can then be written as

$$\begin{aligned} \mathrm{d}s^2 =& \, 4\left((\mathrm{d}\rho_1)^2 + (\mathrm{d}\rho_2)^2 + \frac{1}{4}\left(\rho_1^2 + \rho_2^2\right)\left((\mathrm{d}\Theta)^2 + (\mathrm{d}\phi)^2\right) + \frac{1}{2}\left(\rho_1^2 - \rho_2^2\right)\mathrm{d}\Theta\mathrm{d}\phi\right) + \\ &+ 4\,(\mathrm{d}|\tau|)^2 + \frac{1}{|\tau|^2}\left(\mathrm{d}(R\sin\phi)\right)^2 - \frac{1}{|\tau|^2}\left(\cos\phi(\rho_1\mathrm{d}\rho_2 - \rho_2\mathrm{d}\rho_1) + R\sin\phi\mathrm{d}\Theta\right)^2 , \end{aligned} \tag{3.6}$$

where we have introduced the variables

$$\begin{aligned} \Theta &:= \frac{1}{2}(\theta_1 + \theta_2)\,, \\ \phi &:= \frac{1}{2}(\theta_1 - \theta_2)\,, \\ R &:= \rho_1\rho_2 \,. \end{aligned} \tag{3.7}$$

For the potential, we essentially follow the work of Dorey et al. [18]. This requires writing $(D_i\Phi)^2$ in terms of the ADHM data and extracting the relevant parts in the large distance approximation. The result is

$$V = v^2\left(\left(\rho_1^2 + \rho_2^2\right) - \frac{1}{|\tau|^2}R^2\sin^2\phi\right) . \tag{3.8}$$

There are of course many terms at order $1/|\tau|^4$ which we have ignored here.

The metric (3.6) admits a Killing vector $\partial/\partial\Theta$. As expected from the requirement that our model can be supersymmetrised, one now indeed finds that the norm of this Killing vector equals (up to an overall constant) the potential,

$$\left|\frac{\partial}{\partial\Theta}\right|^2 = \frac{V}{v^2}\,. \tag{3.9}$$

Because the potential gradient contracted with the Killing vector vanishes, i.e.

$$\frac{\partial V(x)}{\partial\Theta} = 0\,, \tag{3.10}$$

there is a conserved quantity associated to this Killing vector; it is

$$\begin{aligned} L &= g_{\mu\Theta}\dot{x}^\mu \\ &= \dot{\Theta}\Big((\rho_1^2+\rho_2^2-\frac{1}{|\tau|^2}\rho_1^2\rho_2^2\sin^2\phi\Big)+(\rho_1^2-\rho_2^2)\dot{\phi}-\frac{1}{|\tau|^2}\cos\phi\sin\phi(\rho_1\dot{\rho}_2-\rho_2\dot{\rho}_1) \\ &= \frac{\dot{\Theta}V}{v^2}+(\rho_1^2-\rho_2^2)\dot{\phi}-\frac{1}{|\tau|^2}\cos\phi\sin\phi(\rho_1\dot{\rho}_2-\rho_2\dot{\rho}_1)\,. \end{aligned} \tag{3.11}$$

The effective action and the corresponding equations of motion can now be written down easily using the metric (3.6) and the potential (3.8). We will refrain from spelling these out, as they are rather lengthy, but some parts of them will be displayed later when we discuss the actual dynamics.

3.2 Validity of the approximation

Before we analyse the orbits of the two-dyon system, let us make a few comments on the validity of the approximation. We have already noted that we are working in the limit in which v and the velocities are small, but since v and $\dot{\Theta}$ are not dimensionless they have to be compared to typical scales in the problem. The limit in which our approximation holds is therefore

$$\begin{aligned} (\rho(t)\,v)^2 &\ll 1\,, & (\dot{\rho}(t))^2 &\ll 1\,, \\ \Big(\rho(t)\,\dot{\Theta}(t)\Big)^2 &\ll 1\,, & (\dot{\rho}(t))^2 &\ll 1\,, \\ \Big(\rho(t)\,\dot{\phi}(t)\Big)^2 &\ll 1\,, & \left(\frac{\rho^2}{|\tau|^2}\right)^2 &\ll 1\,. \end{aligned} \tag{3.12}$$

In these expressions, ρ denotes any typical size parameter in the problem. The last condition could in principle be relaxed if one were brave enough to include all the higher order $1/|\tau|$ corrections into the metric and the potential calculation. The other conditions are intrinsic to the moduli-space approximation and cannot easily be avoided.

3.3 Analytical and numerical analysis of the orbits

We are now ready to analyse the solutions to the equations of motion for the two-instanton system. From sections 2.2 and 2.3 we have learnt that there are two different types of geodesics for the *single* dyonic instanton: the BPS dyonic instanton determined by

$$\rho_i = \text{constant}\,, \qquad \dot{\theta}_i = v\,, \tag{3.13}$$

and the oscillating dyonic instanton, given by the more general solution

$$\rho_i^2 = A\sin(v(t+t_0)) + B\,, \qquad \dot\theta_i = \frac{L}{\rho_i^2}\,,$$
$$B \equiv \sqrt{\frac{L^2}{v^2} + A^2}\,. \tag{3.14}$$

In this section we will analyse the interaction and scattering processes which as asymptotic states have: (i) two BPS dyonic instantons (3.13) or (ii) two oscillating dyonic instantons (3.14). As should be clear from the formulae for the metric and potential, we are unable to give a completely analytic treatment of the dynamics. The main qualitative features can however be extracted with some help of numerical integration.[9]

(i) two BPS dyonic instantons. The first check to perform is to verify that the double dyonic instanton system at rest remains static at any finite separation. This is required, since (3.13) describes a BPS object, and hence it satisfies the no-force condition. One indeed easily verifies that two copies of solution (3.13) for arbitrary values of the instanton radii and arbitrary initial values of angle orientation θ_i ($i = 1, 2$) solve the full equations of motion for the metric (3.6) with the potential (3.8).

When we kick the dyonic instantons towards each other, the dynamics depends crucially on the value of the relative orientation ϕ in the gauge group. For $\phi = 0$ or $\phi = \pi$, the sizes remain constant and the group rotations remain undisturbed (this behaviour is in contrast to that of ordinary 1/2-BPS dyons, which when moving feel a force due to the difference in the velocity dependence of the scalar and vector forces). For the intermediate value $\phi = \pi/2$ the two objects behave identically, with the radii slowly increasing as they approach each other. In between these values, the behaviour of the radii is more complicated.

(ii) two oscillating instantons. Let us first consider two oscillating instantons which (when the instantons are far apart) have identical ρ oscillations. The equation of motion for the separation τ following from (3.6) and (3.8) is given by

$$-8\,\ddot\tau - \frac{2v^2}{|\tau|^3}(\rho_1\rho_2)^2\sin^2\phi - \frac{2}{|\tau|^3}\Big(\sin\phi(\rho_1\dot\rho_2 + \rho_2\dot\rho_1) + \rho_1\rho_2\,\dot\phi\,\cos\phi\Big)^2 + \\ + \frac{2}{|\tau|^3}\Big(\cos\phi(\rho_1\dot\rho_2 - \rho_2\dot\rho_1) + \rho_1\rho_2\,\dot\Theta\,\sin\phi\Big)^2 = 0\,. \tag{3.15}$$

From this expression one notes that evolution with $\phi(t) = 0$ and identical radii, $\rho_1(t) = \rho_2(t)$, leads to a vanishing force and hence $\tau =$ constant (inspection of the equation of motion for ϕ shows that $\phi(t) = 0$ can indeed be imposed for arbitrary times). So we see that even though two identical and synchronised oscillating dyonic instantons are not strict BPS states in the full theory, they behave as BPS objects in our approximation. The oscillating motion found in section 2.3 solves the equations of motion of the effective action even when the solitons are at finite separation.

[9]The computer programs used to produce these numerical solutions can be obtained from http://www.damtp.cam.ac.uk/user/kp229/di.html

JHEP12(2001)032

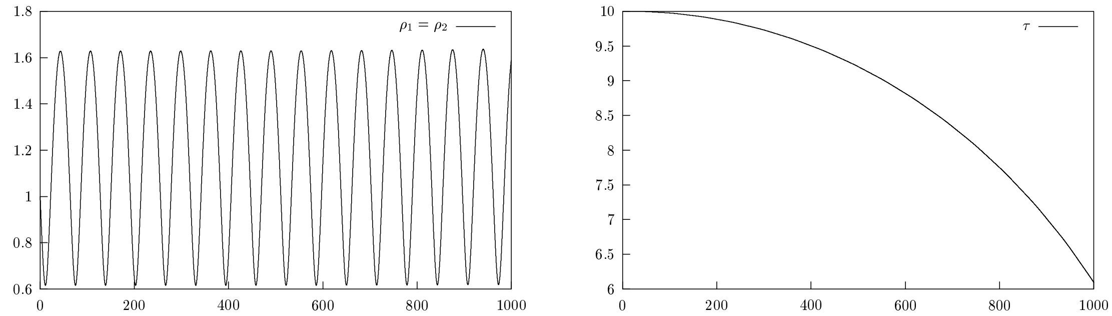

Figure 3: Solution for which $\theta_1 \neq \theta_2$, but the radii still satisfy $\rho_1 = \rho_2$. This is only possible for $\phi = \pi/2$ for all times, which implies that the group rotation of the two solitons is synchronised. The solitons feel an attractive force. The other initial values are $\rho_1 = \rho_2 = 1$, $\tau = 10$, $\dot{\rho}_1 = \dot{\rho}_2 = -0.05$, $A = 1.118$.

More interesting dynamics occurs when one takes the relative angle for the gauge embedding of the instantons to be non-zero ($\theta_1 \neq \theta_2$, i.e. $\phi \neq 0$) while keeping the ρ amplitudes the same. In this case one finds an attractive force between the instantons for all initial values of ϕ. A typical orbit is depicted in figure 3. This behaviour is difficult to extract from the equation of motion (3.15) directly (due to the non-trivial oscillations of the radii). However, we will see in the next section how an effective model can be derived that predicts this behaviour.

Another, more complicated set of geodesics appears in case one has two asymptotic dyonic instantons which oscillate with the same ρ amplitudes but with opposite phases. Depending on the initial value of the angle ϕ one finds attractive or, in contrast to the situation before, repulsive behaviour. On top of this long time scale motion, there are high-frequency oscillations in τ which were not present in the previous (equal phase) situation. The borderline case, where only high-frequency oscillations remain, is hard to determine, but we will see that again an effective model can come to the rescue.

Finally, when the condition of equal radii is dropped, the full structure of the force between the solitons becomes visible. Attractive, repulsive and oscillatory behaviour of τ are again present, but the equations are generically too difficult to analyse analytically. Typical orbits of this type, obtained using numerical integration, are depicted in figures 4 and 5.

3.4 Effective action for two oscillating dyonic instantons

Although the moduli space approach significantly simplifies the analysis of soliton dynamics by getting rid of high-energy modes, one can often still not do without numerical help when solving for the low-energy behaviour. In the case of 1/2-BPS dyons there is not much else one can do, but for the 1/4-BPS solitons under consideration here, we have seen that even in the low-energy regime there is sometimes a separation of different scales. For instance, the attractive and repulsive orbits discussed in the previous section exhibit high-frequency oscillation set by the scale of the instantons, and low-frequency dynamics set by the scale of separation. In the present section, we will explain how one can simplify the analysis in certain cases, by integrating out all but the lowest of the remaining modes.

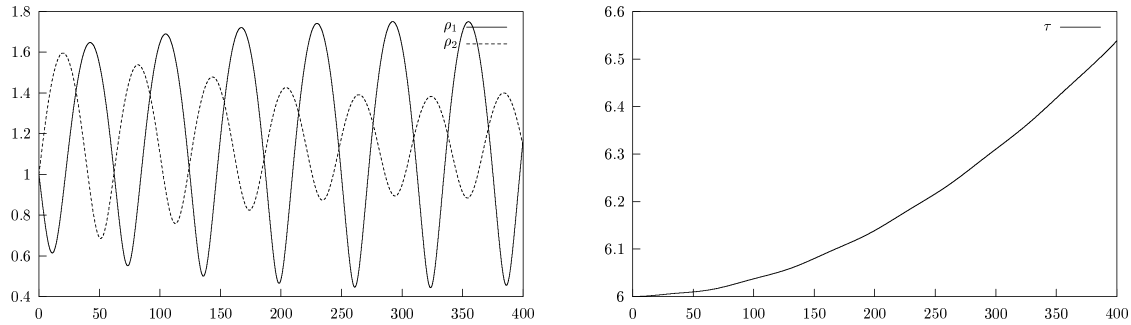

Figure 4: Generic double soliton solution of the "repulsive" type. The initial data are $\tau = 6$, $\rho_1 = \rho_2 = 1$, $\dot{\rho}_1 = -\dot{\rho}_2 = -0.05$ and $\dot{\phi} = 0.7$. There are oscillations on top of the curve of $\tau(t)$, but they are too small to be clearly visible in this plot.

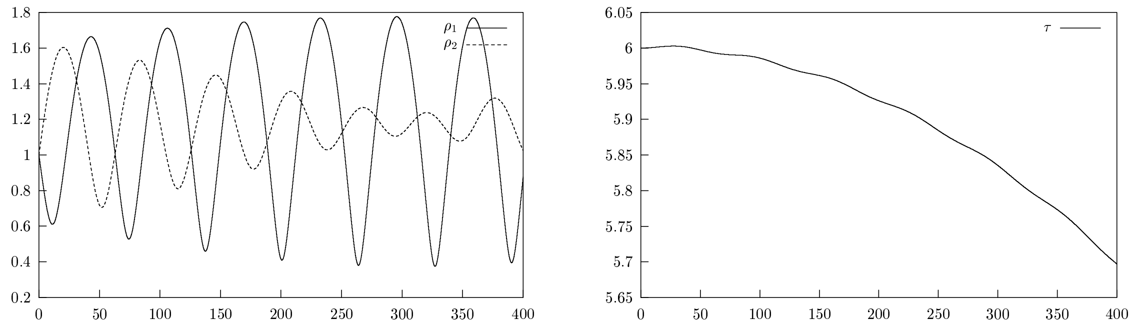

Figure 5: Generic double soliton solution of the "attractive" type. The initial data are similar to those of 5 with the difference that now $\dot{\phi} = 0.3$. Again, there are oscillations on top of the τ curve which are very small.

Let us first consider the case of two synchronised oscillating instantons, $\rho_1(t) = \rho_2(t)$ with arbitrary ϕ and θ. As we have seen in the numerical analysis, this system always exhibits attractive behaviour. Strictly speaking the amplitudes of ρ_1 and ρ_2 are not equal in time, and not even constant (the only case in which this is true is $\phi = \pi/2$, given in figure 3). However, since the rate of amplitude decrease is much smaller than the rate of the decrease of the instanton separation τ, one may approximate the instanton oscillation during motion with the free one (3.14). Numerical analysis also shows that the relative phase ϕ oscillates with increasing amplitude. However as a first approximation we will take ϕ to be a constant during motion.

So we proceed by inserting the $\rho_1(t) = \rho_2(t)$ given by (3.14) and $\theta = \text{const.}, \phi = \text{const.}$ into the effective action derived from (3.6) and (3.8), and average over one period to arrive at

$$S_{\text{eff}} = \int \mathrm{d}t \left(4\,\dot{\tau}^2 + \frac{2\,v^2 A^2\,\sin^2\phi}{\tau^2} \right). \tag{3.16}$$

As expected we see that potential is attractive for all values of the angle ϕ. The solution to the equation of motion is easily found,

$$\tau = \sqrt{\tau_0^2 - \frac{v^2 A^2 \sin^2(\phi)}{2\,\tau_0^2} t^2}\,. \tag{3.17}$$

From this expression we can see that the Higgs expectation value v does not appear in the separation of the scales: the time scales for ρ and τ motion are $\omega_\rho = v$ and $\omega_\tau = vA/\tau^2$, respectively. The agreement of the low-frequency mode approximation with the full dynamics is excellent: when the analytical result is compared with the numerical one of figure 3, the errors are on the order of the width of the curve.

It is important to stress that even though we have integrated out high-frequency behaviour associated to the non-zero modes, this is by no means equivalent to ignoring them. While the latter corresponds to keeping the radii fixed and leads to a vanishing force (as discussed in part (i) of the previous section) averaging over the non-zero modes leads to an attractive force.

Next we consider two oscillating instantons with opposite phases

$$\begin{aligned} \rho_1^2 &= A\sin(vt) + B\,, \qquad \dot\theta_1 = \frac{L}{\rho_1^2}\,, \\ \rho_2^2 &= -A\sin(vt) + B\,, \qquad \dot\theta_2 = \frac{L}{\rho_2^2}\,. \end{aligned} \tag{3.18}$$

By integrating the equation for $\dot\phi = \frac{1}{2}(\dot\theta_1 - \dot\theta_2)$ one obtains

$$\tan(\phi - \phi_0) = \frac{Av}{L}\cos(vt)\,. \tag{3.19}$$

Inserting (3.18) and (3.19) into the action constructed from the metric (3.6) and the potential (3.8), one then finds

$$\begin{aligned} S = \int 4(\dot\tau)^2 + \frac{1}{\tau^2}\frac{L^2}{B^2 - A^2\sin^2(vt)}\frac{1}{1+\tan^2\phi} \times \\ \times \left(A^2\sin^2(vt)\left(1 + \frac{Av}{L}\cos(vt)\tan(\phi)\right)^2 - B^2\left(\tan(\phi) - \frac{Av}{L}\cos(vt)\right)^2\right) + \cdots\,, \end{aligned} \tag{3.20}$$

where dots corresponds to terms which after averaging produce constant terms. By averaging the terms proportional to $1/\tau^2$ one obtains again the effective force between two oscillating instantons. These integrations are rather tedious to do analytically, so we have restricted ourselves to a numerical integration for various values of the ϕ_0 parameter. One finds that the effective action (3.20) is indeed able to predict the attractive, repulsive and oscillatory phases as obtained in the full numerical analysis of section 3.3. Again, we should stress that integrating out the non-zero modes is crucially different from simply ignoring them, and leads to much richer dynamics.

4. Discussion, conclusions and outlook

In this paper we have exhibited several new features that arise when the moduli space method is extended from 1/2-BPS to 1/4-BPS soliton dynamics. The main ingredient

JHEP12(2001)032

responsible for these new features is the appearance of a potential, which we have explained from the point of view of the underlying *approximately static* uncharged soliton. This approach has naturally led us to consider dynamics for which the potential energy is not constant during evolution. The non-zero mode excitations can be taken into account in a way that is consistent with the moduli space approximation, and provide genuinely new types of behaviour. A second new feature is the appearance of different scales even in the low-energy regime. This is again intimately related to the presence of the potential: orbits which involve motion up and down the potential "superposed" with motion in the flat directions exhibit such a scale separation. We have shown how one can integrate out the high-frequency modes among the excitations on moduli space, and arrive at a new effective action for the remaining modes. We were able to illustrate these features on the very explicit example of two dyonic instantons, as we managed to write down an explicit expression for the metric and the potential of this model.

For future research, several interesting questions remain open. Firstly, the analysis of the dynamics is far from complete (we have only considered head-on motion of the two solitons) and one may want study more complicated orbits. Given the complexity of the metric, we have also not been able to analyse full head-on scattering that brings the solitons very close together. A particularly interesting question is to understand the effect of the potential on the 90° scattering process familiar from dynamics of other solitons.

On a more fundamental level, it would be interesting to study the dynamical properties of the full supersymmetric extension of our model, as well as the quantisation of it. Previous studies in this direction (see for instance Bak et al. [32]) have focused on the kinematical properties. Finally, there are other systems for which the dynamics under influence of the potential can be analysed explicitly. While most of the SU(3) monopole systems have a very complicated metric (see Houghton and Lee [33] for details), the $(1,1)$ system is amenable to direct analytical analysis. Work in this direction is in progress and will be published shortly.

Acknowledgments

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A. Appendix: technical details

A.1 Two-instanton metric

The crucial ingredients for the computation of the metric on the two-instanton moduli space can be found in Osborn [14]. We will here summarise the relevant details in a coordinate system that is more suitable for our purposes; the reader is referred to [14] for a detailed explanation of the formalism.

The square root Δ of the ADHM projection matrix will be parametrised as

$$\Delta = a + bx \tag{A.1}$$

with

$$a = \begin{pmatrix} v_1 & v_2 \\ \rho+\tau & \sigma \\ \sigma & \rho-\tau \end{pmatrix}, \qquad b = \begin{pmatrix} 0 & 0 \\ \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}. \tag{A.2}$$

The normalisation parameter ν appearing in $b^\dagger b = \nu\mathbb{1}$ is therefore equal to one. The projector P_∞ becomes

$$P_\infty = \lim_{x\to\infty}(1 - b\nu^{-1}b^\dagger) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \text{so that} \qquad (1+P_\infty) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{A.3}$$

We are now ready to solve the non-linear ADHM constraint $a^\dagger a = \mu\mathbb{1}$. It leads to

$$\bar{v}_1 v_2 - \bar{v}_2 v_1 = 2(\bar{\sigma}\tau - \bar{\tau}\sigma)\,, \tag{A.4}$$

which is necessary to make the off-diagonal components of μ vanish (they are manifestly not proportional to the unit matrix in SU(2) space). This allows us to solve for σ in terms of the other moduli:

$$\sigma = \tau\frac{\bar{v}_2 v_1 - \bar{v}_1 v_2}{4|\tau|^2} + \alpha\tau =: \tau\left(\frac{\Lambda}{4|\tau|^2} + \alpha\right). \tag{A.5}$$

We will set $\alpha = 0$ in order to be able to use the interpretation of the moduli as given by Dorey et al. [18]. In order to make contact with the work of Osborn [14], we have to find a real function λ such that σ above satisfies

$$\bar{v}_2 v_1 - 2\bar{\tau}\sigma = 2\lambda|\tau|^2\,. \tag{A.6}$$

One finds that

$$\lambda = \frac{\bar{v}_2 v_1 + \bar{v}_1 v_2}{2|\tau|^2}\,. \tag{A.7}$$

We will see in a moment that this is proportional to $\delta^a{}_b$ in spinor space. It is also useful to have a compact expression for Λ,

$$\Lambda = 4\, v_2^m v_1^n \bar{\sigma}_{mn} = -\bar{\Lambda}\,. \tag{A.8}$$

The metric for the zero-modes can be obtained from Osborn [14, (4.14)],

$$\langle C', C\rangle = \underbrace{\langle \delta' a, \delta a\rangle}_{:=\langle C',C\rangle_1} - \underbrace{4k'kN_A^{-1}}_{:=\langle C',C\rangle_2}, \tag{A.9}$$

where the inner product of the first term is given by

$$\langle \delta' a, \delta a\rangle = \mathrm{tr}\left(\delta' a^\dagger(1+P_\infty)\delta a\right). \tag{A.10}$$

The first term in (A.9) leads to a diagonal contribution to the metric, which is easy to see by computing

$$\begin{aligned} &\delta' a^\dagger(1+P_\infty)\delta a = \\ &= \begin{pmatrix} 2\delta'\bar{v}_1\delta v_1 + (\delta'\bar{\rho}+\delta'\bar{\tau})(\delta\rho+\delta\tau) + \delta'\bar{\sigma}\delta\sigma & \cdots \\ \cdots & 2\delta'\bar{v}_2\delta v_2 + (\delta'\bar{\rho}-\delta'\bar{\tau})(\delta\rho-\delta\tau) + \delta'\bar{\sigma}\delta\sigma \end{pmatrix}. \end{aligned} \tag{A.11}$$

The trace then eliminates the $\delta\rho\delta\tau$ cross-term, and the result for the first term in the metric is

$$\langle C', C\rangle_1 = 2\ \mathrm{tr}_2\left(\delta'\bar{v}_1\delta v_1 + \delta'\bar{v}_2\delta v_2 + \delta'\bar{\rho}\delta\rho + \delta\bar{\tau}\delta\tau + \delta'\bar{\sigma}\delta\sigma\right). \tag{A.12}$$

The final step consists of eliminating the σ in the last term. The variations of $\bar{\sigma}$ and σ are

$$\begin{aligned} \delta'\bar{\sigma} &= \delta'\bar{\Lambda}\bar{\tau}\frac{1}{4|\tau|^2} + \bar{\Lambda}\delta'\bar{\tau}\frac{1}{4|\tau|^2} - \bar{\Lambda}\bar{\tau}\delta'(\tau\bar{\tau})\frac{1}{4|\tau|^4}\,, \\ \delta\sigma &= \delta\tau\Lambda\frac{1}{4|\tau|^2} + \tau\delta\Lambda\frac{1}{4|\tau|^2} - \tau\Lambda\delta(\tau\bar{\tau})\frac{1}{4|\tau|^4}\,. \end{aligned} \tag{A.13}$$

The product thus leads to nine different terms for which we have to work out the conversion of quaternions to vectors.

$$\begin{aligned} (\delta'\bar{\sigma})(\delta\sigma) &= \delta'\bar{\Lambda}\bar{\tau}\delta\tau\Lambda\frac{1}{16|\tau|^4} + \delta'\bar{\Lambda}\delta\Lambda\frac{1}{16|\tau|^2} - \delta'\bar{\Lambda}\Lambda\delta(\tau\bar{\tau})\frac{1}{16|\tau|^4} + \\ &+ \bar{\Lambda}\delta'\bar{\tau}\delta\tau\Lambda\frac{1}{16|\tau|^4} + \bar{\Lambda}\delta'\bar{\tau}\tau\delta\Lambda\frac{1}{16|\tau|^4} - \bar{\Lambda}\delta'\bar{\tau}\tau\Lambda\delta(\tau\bar{\tau})\frac{1}{16|\tau|^6} - \\ &- \bar{\Lambda}\bar{\tau}\delta'(\tau\bar{\tau})\delta\tau\Lambda\frac{1}{16|\tau|^6} - \bar{\Lambda}\delta'(\tau\bar{\tau})\delta\Lambda\frac{1}{16|\tau|^4} + \bar{\Lambda}\Lambda\delta'(\tau\bar{\tau})\delta(\tau\bar{\tau})\frac{1}{16|\tau|^6}\,. \end{aligned} \tag{A.14}$$

In order to limit the number of σ matrices appearing in the trace, it is convenient to rewrite $\delta(\tau\bar{\tau})$ right from the beginning in vector notation as

$$\delta(\tau\bar{\tau}) = 2\mathbb{1}_2(\tau\cdot\delta\tau)\,. \tag{A.15}$$

In the main text we ignore all $1/|\tau|^4$ terms, so there is only one term remaining,

$$\begin{aligned} \frac{1}{8}\,\mathrm{tr}(\delta'\bar{\Lambda}\delta\Lambda) &= (\delta'v_2\cdot\delta v_2)|v_1|^2 + (v_1\cdot\delta'v_1)(v_2\cdot\delta v_2) + (v_1\cdot\delta v_1)(v_2\cdot\delta'v_2) + \\ &+ (\delta'v_1\cdot\delta v_1)|v_2|^2 - (\delta'v_2\cdot v_1)(\delta v_2\cdot v_1) - (\delta'v_1\cdot v_2)(\delta v_1\cdot v_2) - \\ &- (v_2\cdot v_1)(\delta'v_1\cdot\delta v_2) - (v_2\cdot v_1)(\delta v_1\cdot\delta'v_2) - \\ &- \left(v_2^m\delta'v_1^n\delta v_2^k v_1^l + v_2^k\delta v_1^l\delta'v_2^m v_1^n\right)\epsilon_{mnkl}\,. \end{aligned} \tag{A.16}$$

Now we focus on the second part, $\langle C', C\rangle_2$. For this we have to compute

$$a^\dagger\delta a - (a^\dagger\delta a)^T = K = \mathbb{1}_2\otimes\begin{pmatrix} 0 & k \\ -k & 0\end{pmatrix}. \tag{A.17}$$

For the matrix element K_{12} one finds

$$K_{12} = \bar{v}_1\delta v_2 - \bar{v}_2\delta v_1 + 2\bar{\tau}\delta\sigma - 2\bar{\sigma}\delta\tau\,. \tag{A.18}$$

Inserting the solution (A.5) of the ADHM constraint and using (A.15), this becomes

$$K_{12} = \frac{1}{2}(\bar{v}_1\delta v_2 + \delta\bar{v}_2 v_1 - \bar{v}_2\delta v_1 - \delta\bar{v}_1 v_2) - \frac{1}{2}\left(\frac{\bar{\Lambda}}{|\tau|^2}\bar{\tau}\delta\tau + \delta\bar{\tau}\tau\frac{\Lambda}{|\tau|^2}\right). \tag{A.19}$$

As expected this is indeed proportional to the identity matrix in SU(2): the terms that only have v_i are proportional to $\bar{\sigma}_m\sigma_n + \bar{\sigma}_n\sigma_m$, while the other ones are proportional to

$\bar\sigma_{mn}\bar\sigma_{kl}$, both of which (by (A.29) and (A.26), respectively) are proportional to the identity. Rewriting k in vector notation, we find

$$k = (v_1 \cdot \delta v_2) - (v_2 \cdot \delta v_1) - \frac{1}{|\tau|^2}\Big(\epsilon_{mnkl} v_2^m v_1^n \tau^k \delta\tau^l - (v_2\cdot\tau)(v_1\cdot\delta\tau) - \\ - (v_1\cdot\tau)(v_2\cdot\delta\tau)\Big)\,. \tag{A.20}$$

From this we now have to compute $k'k(N_A)^{-1}$. This requires the matrix N_A, which is given in (3.30) of Osborn's paper,

$$N_A = |v_1|^2 + |v_2|^2 + 4(|\tau|^2 + |\sigma|^2) = |v_1|^2 + |v_2|^2 + 4|\tau|^2 + |\bar v_2 v_1 - \bar v_2 v_1|^2\,. \tag{A.21}$$

The result for $k'kN_A^{-1}$ can now easily be read off.

A.2 Conventions and sigma algebra

This appendix collects some useful expressions for euclidean sigma matrices (indices m, n, ... take values $0, 1, 2, 3$ but the zeroth direction has positive signature too). The basic definitions in terms of the Pauli matrices τ^i are

$$\begin{aligned}
\sigma^m &:= (\mathbb{1}, i\tau^i)\,,\\
\bar\sigma^m &:= (\mathbb{1}, -i\tau^i)\,,\\
\sigma^{mn} &:= \frac{1}{4}(\sigma^m\bar\sigma^n - \sigma^n\bar\sigma^m)\,,\\
\bar\sigma^{mn} &:= \frac{1}{4}(\bar\sigma^m\sigma^n - \bar\sigma^n\sigma^m)\,,\\
\sigma^{mnp} &:= \frac{1}{2}(\sigma^m\bar\sigma^n\sigma^p - \sigma^p\bar\sigma^n\sigma^m)\,,\\
\bar\sigma^{mnp} &:= \frac{1}{2}(\bar\sigma^m\sigma^n\bar\sigma^p - \bar\sigma^p\sigma^n\bar\sigma^m)\,.
\end{aligned}\tag{A.22}$$

The sigma matrices satisfy the algebra

$$\sigma_l\bar\sigma_m = 2\,\sigma_{lm} + \delta_{lm}\,, \qquad \bar\sigma_l\sigma_m = 2\,\bar\sigma_{lm} + \delta_{lm}\,. \tag{A.23}$$

and one can derive analogues for products of more sigma matrices.

$$\begin{aligned}
\sigma_n\bar\sigma_{kl} &= \frac{1}{2}\sigma^{nkl} + \delta^{n[k}\sigma^{l]} = \frac{1}{2}\epsilon^{nklq}\sigma_q + \delta_{n[k}\sigma_{l]}\,,\\
\sigma_m\bar\sigma_{nkl} &= \epsilon_{mnkl} - 2\epsilon_{nklq}\sigma_{mq}\,,\\
\sigma_{mn}\sigma_{kl} &= -\frac{1}{4}\delta_{mk}\delta_{nl} + \frac{1}{4}\delta_{ml}\delta_{nk} + \frac{1}{4}\epsilon^{mnkl}\,,\\
\bar\sigma_{mn}\bar\sigma_{kl} &= -\frac{1}{4}\delta_{mk}\delta_{nl} + \frac{1}{4}\delta_{ml}\delta_{nk} - \frac{1}{4}\epsilon^{mnkl}\,.
\end{aligned}\tag{A.24}$$

The epsilon symbols (which satisfy $\epsilon_{0123} = 1$) arise because of the following duality flip relations,

$$\begin{aligned}
\sigma_{mn} &= -\frac{1}{2}\epsilon_{mnpq}\sigma_{pq}\,,\\
\bar\sigma_{mn} &= \frac{1}{2}\epsilon_{mnpq}\bar\sigma_{pq}\,,\\
\sigma^{mnp} &= \epsilon^{mnpq}\sigma_q\,,\\
\bar\sigma^{mnp} &= -\epsilon^{mnpq}\bar\sigma_q\,.
\end{aligned}\tag{A.25}$$

Finally, one can of course trace these expressions to arrive at

$$\mathrm{tr}_2\,(\sigma^{mnp}) = 2\epsilon^{mnp0}\,, \tag{A.26}$$

$$\mathrm{tr}_2\,(\sigma_{mn}\sigma_{kl}) = -\frac{1}{2}\delta_{mk}\delta_{nl} + \frac{1}{2}\delta_{ml}\delta_{nk} + \frac{1}{2}\epsilon^{mnkl}\,, \tag{A.27}$$

$$\mathrm{tr}_2\,(\bar\sigma_{mn}\bar\sigma_{kl}) = -\frac{1}{2}\delta_{mk}\delta_{nl} + \frac{1}{2}\delta_{ml}\delta_{nk} - \frac{1}{2}\epsilon^{mnkl}\,, \tag{A.28}$$

$$\mathrm{tr}_2\,(\sigma_m\bar\sigma_n + \sigma_n\bar\sigma_m) = 4\delta_{mn}\,. \tag{A.29}$$

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